LINEAR ALGEBRA, MMATH MID SEMESTRAL EXAMINATION, SEPTEMBER 2009

Please attempt all questions. If you use any theorem which has been taught in class in your answer, then please state it in full and clearly. All questions carry equal marks. Total Marks - 50.

Question 1 Let F be any field, and let $A \in M_n(F)$ be a $n \times n$ matrix over F which is not invertible. Prove that there exists $B \in M_n(F)$ with $B \neq 0$, such that $BA = 0$.

Question 2 Let F be any field. Prove that the centre of $GL_n(F)$ is the subgroup $\{\lambda.I_n | \lambda \in F\}$, where I_n is the $n \times n$ identity matrix.

Question 3 Let V be the set of all sequences of real numbers $\{a_i\}_{i=1}^{\infty}$, which have the property that $a_i = 0$ for all but finitely many indices i. V is a real vector space under coordinate-wise addition and scalar multiplication. Prove that V is not finite dimensional. Are V and $\mathbb{R}[x]$ isomorphic as real vector spaces?

Question 4 Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear operator such that $T^2 = T$. Prove that, either $T = 0$, or $T = Id$, or there exists an ordered basis B of $\mathbb{R}^2$ such that the matrix of T with respect to B is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

Question 5 Let V be a vector space over the field F. Let $V^* = Hom_F(V, F)$ denote the dual vector space of V, and let $f, g \in V^*$. Suppose, for any $v \in V$, $f(v) = 0$ implies that $g(v) = 0$. Prove that $g = \lambda.f$ for some $\lambda \in F$.

Question 6 Let V be a finite dimensional vector space over a field F. Let T be a linear operator on V which has two linearly independent eigenvectors with the same eigenvalue λ. Is it true that λ is a multiple root of the characteristic polynomial of T?